Locating All Minima of a Smooth Function Without Access to its Derivative

Jeffrey Larson   Stefan Wild

Argonne National Laboratory

July 16, 2015

Motivation

- We want to identify distinct, "high-quality", local minimizers of

$$\begin{aligned} \text{minimize}\, f(x) \\ l \leq x \leq u \\ x \in \mathbb{R}^n \end{aligned}$$

- High-quality can be measured by more than the objective.

Motivation

- We want to identify distinct, "high-quality", local minimizers of

$$\begin{aligned} \text{minimize}\, f(x) \\ l \leq x \leq u \\ x \in \mathbb{R}^n \end{aligned}$$

- High-quality can be measured by more than the objective.
- Derivatives of f may or may not be available.

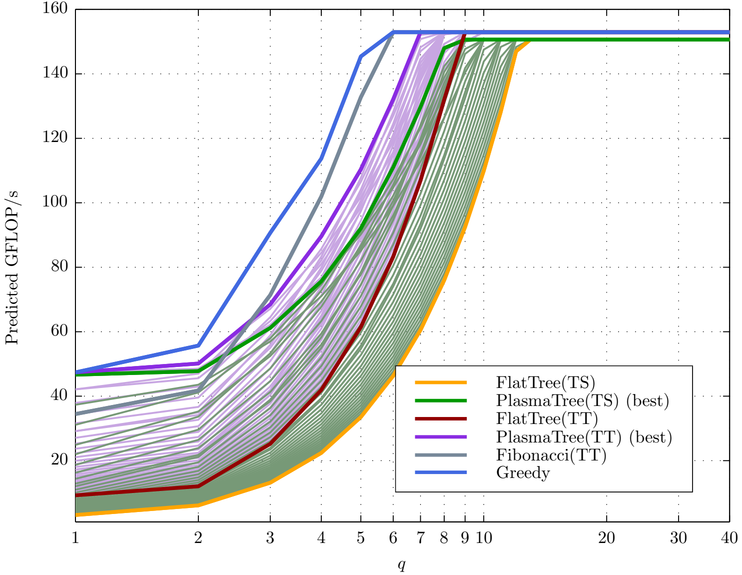
Motivation

- We want to identify distinct, “high-quality”, local minimizers of

$$\begin{aligned} \text{minimize}\, f(x) \\ l \leq x \leq u \\ x \in \mathbb{R}^n \end{aligned}$$

- High-quality can be measured by more than the objective.
- Derivatives of f may or may not be available.
- The simulation f is likely using parallel resources, but it does not utilize the entire machine.

Why concurrency? Tiled QR example



[Bouwmeester, et al., Tiled QR Factorization Algorithms, 2011]

Global optimization is difficult

Theorem (Törn and Žilinskas, *Global Optimization*, 1989)

An algorithm converges to the global minimum of any continuous f on a domain $\mathcal{D}$ if and only if the algorithm generates iterates that are dense in $\mathcal{D}$.

Global optimization is difficult

Theorem (Törn and Žilinskas, *Global Optimization*, 1989)

An algorithm converges to the global minimum of any continuous f on a domain $\mathcal{D}$ if and only if the algorithm generates iterates that are dense in $\mathcal{D}$.

- Either assume additional properties about the problem
  - convex f
  - separable f
  - finite domain $\mathcal{D}$

Global optimization is difficult

Theorem (Törn and Žilinskas, *Global Optimization*, 1989)

An algorithm converges to the global minimum of any continuous f on a domain $\mathcal{D}$ if and only if the algorithm generates iterates that are dense in $\mathcal{D}$.

- Either assume additional properties about the problem
  - convex f
  - separable f
  - finite domain $\mathcal{D}$
- Or possibly wait a long time (or forever)

Global optimization is difficult

Theorem (Törn and Žilinskas, *Global Optimization*, 1989)

An algorithm converges to the global minimum of any continuous f on a domain $\mathcal{D}$ if and only if the algorithm generates iterates that are dense in $\mathcal{D}$.

- Either assume additional properties about the problem
  - convex f
  - separable f
  - finite domain $\mathcal{D}$
  - concurrent evaluations of f
- Or possibly wait a long time (or forever)

Global optimization is difficult

Theorem (Törn and Žilinskas, *Global Optimization*, 1989)

An algorithm converges to the global minimum of any continuous f on a domain $\mathcal{D}$ if and only if the algorithm generates iterates that are dense in $\mathcal{D}$.

- Either assume additional properties about the problem
  - convex f
  - separable f
  - finite domain $\mathcal{D}$
  - concurrent evaluations of f
- Or possibly wait a long time (or forever)

The theory can be more than merely checking that a method generates iterates which are dense in the domain.

Multistart Methods

Given some local optimization routine $\mathcal{L}$:

Algorithm 1: General Multistart

for $k = 1, 2, \ldots$ **do**
  Evaluate f at N points drawn from $\mathcal{D}$
  Start $\mathcal{L}$ at some set (possibly empty) of previously evaluated points

Multistart Methods

Given some local optimization routine $\mathcal{L}$:

Algorithm 1: General Multistart

for $k = 1, 2, \ldots$ **do**
- Evaluate f at N points drawn from $\mathcal{D}$
- Start $\mathcal{L}$ at some set (possibly empty) of previously evaluated points

- Get to use problem specific local optimization routines.
- Possibly multiple levels of parallelism (objective, local method, global method); $\mathcal{L}$ may involve many sequential evaluations of f...

Multistart Methods

Given some local optimization routine $\mathcal{L}$:

Algorithm 1: General Multistart

for $k = 1, 2, \ldots$ **do**

  Evaluate f at N points drawn from $\mathcal{D}$

  Start $\mathcal{L}$ at some set (possibly empty) of previously evaluated points

- Get to use problem specific local optimization routines.
- Possibly multiple levels of parallelism (objective, local method, global method); $\mathcal{L}$ may involve many sequential evaluations of f...

- Which points should start runs?
- If resources are limited, how should points from each run receive priority?

Multistart Methods

Given some local optimization routine $\mathcal{L}$:

Algorithm 1: General Multistart

for $k = 1, 2, \ldots$ **do**
  Evaluate f at N points drawn from $\mathcal{D}$
  Start $\mathcal{L}$ at some set (possibly empty) of previously evaluated points

- Get to use problem specific local optimization routines.
- Possibly multiple levels of parallelism (objective, local method, global method); $\mathcal{L}$ may involve many sequential evaluations of f...

- Which points should start runs?
- If resources are limited, how should points from each run receive priority?

- Ideally, only one run is started for each minima.
- Exploring by sampling. Refining with $\mathcal{L}$.

Multi-Level Single Linkage

Given some local optimization routine $\mathcal{L}$:

Algorithm 2: MLSL

for $k = 1, 2, \ldots$ **do**

Sample f at N random points drawn uniformly from $\mathcal{D}$

Start $\mathcal{L}$ at any sample point x:

- that has yet to start a run
- $\nexists x_i : \|x - x_i\| \leq r_k$ and $f(x_i) < f(x)$

[Rinnooy Kan and Timmer, *Mathematical Programming*, 39(1):57–78, 1987]

Multi-Level Single Linkage

Given some local optimization routine $\mathcal{L}$:

Algorithm 2: MLSL

for $k = 1, 2, \ldots$ **do**

Sample f at N random points drawn uniformly from $\mathcal{D}$

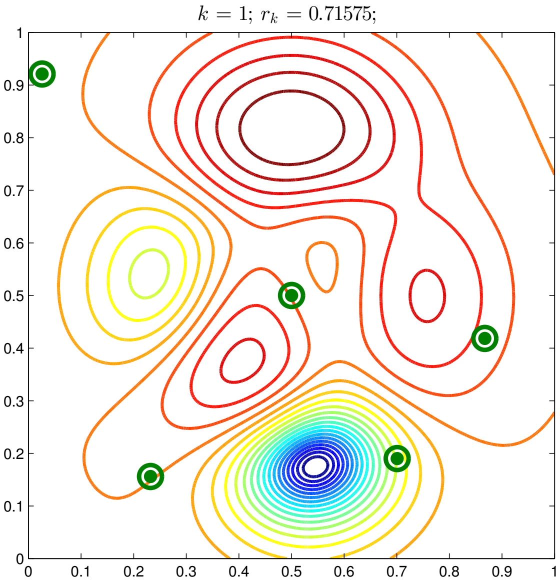
Start $\mathcal{L}$ at any sample point x:

- that has yet to start a run
- $\nexists x_i : \|x - x_i\| \leq r_k$ and $f(x_i) < f(x)$

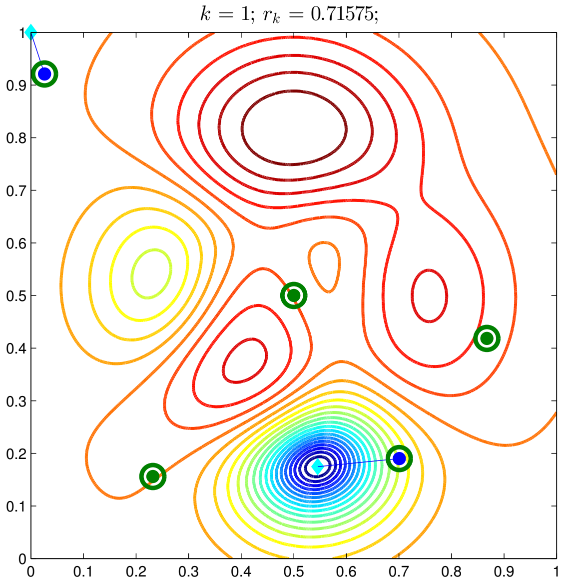
[Rinnooy Kan and Timmer, *Mathematical Programming*, 39(1):57–78, 1987]

- Doesn't naturally translate when evaluations of f are limited
- Ignores some points when deciding where to start $\mathcal{L}$

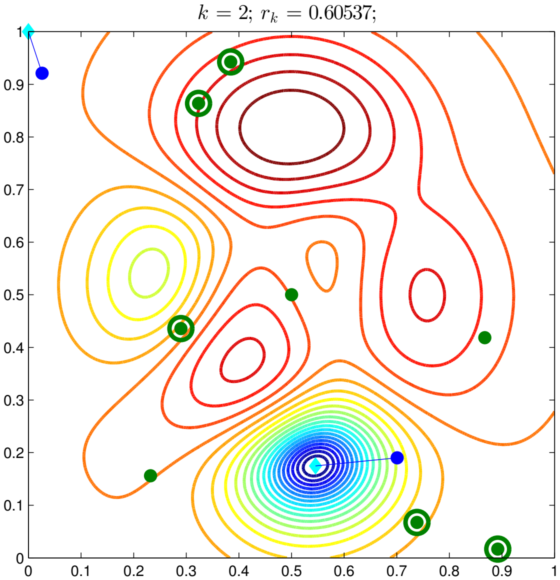
Multi-Level Single Linkage



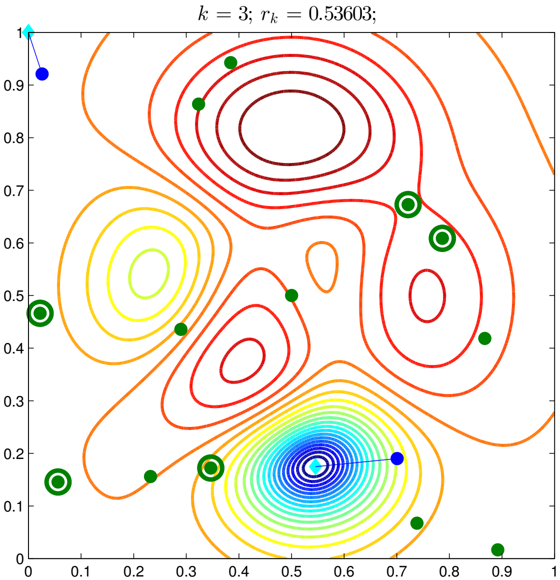
Multi-Level Single Linkage



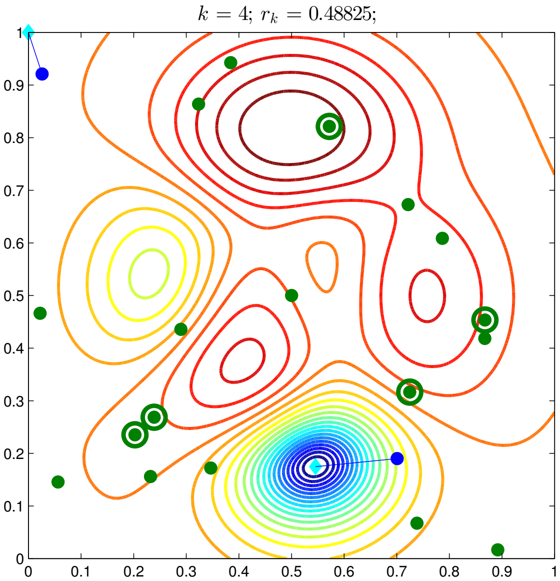
Multi-Level Single Linkage



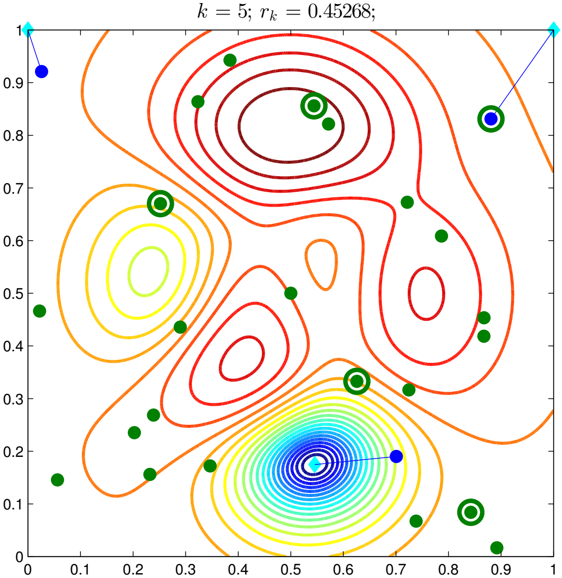
Multi-Level Single Linkage



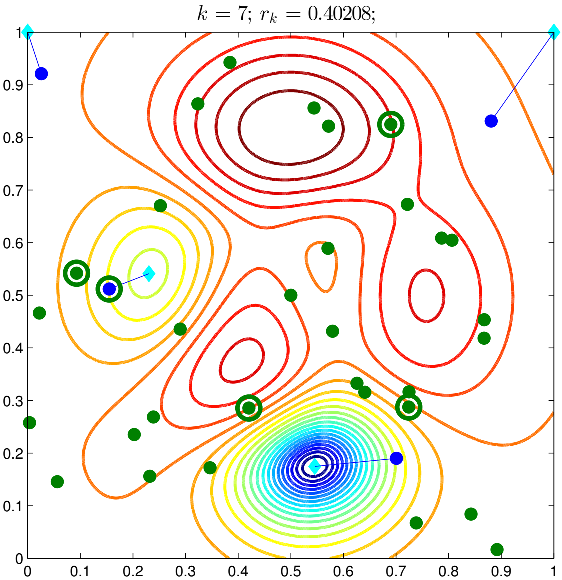
Multi-Level Single Linkage



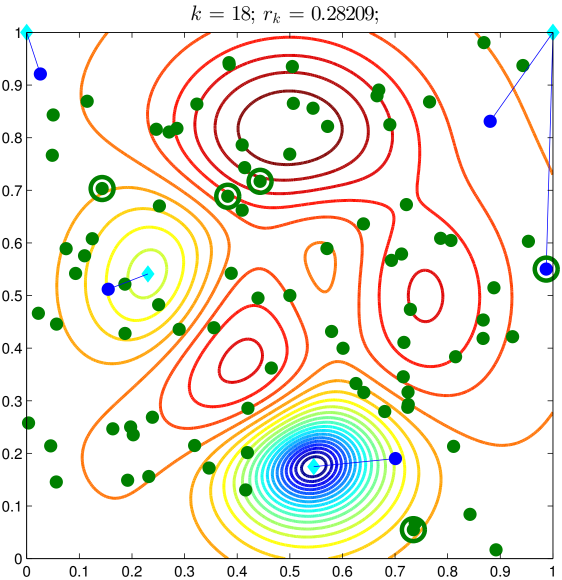
Multi-Level Single Linkage



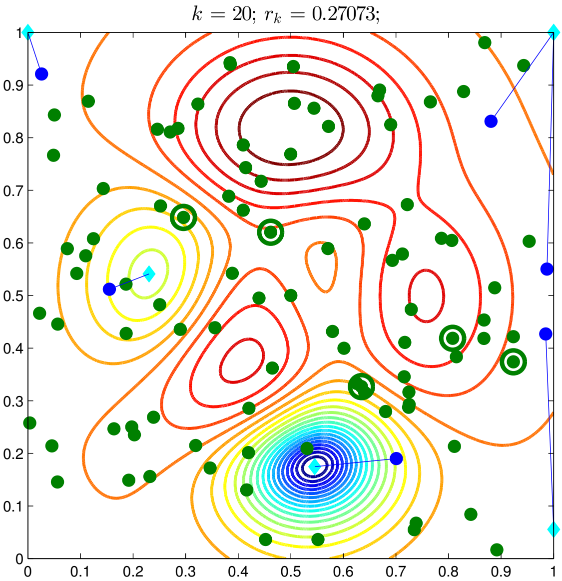
Multi-Level Single Linkage



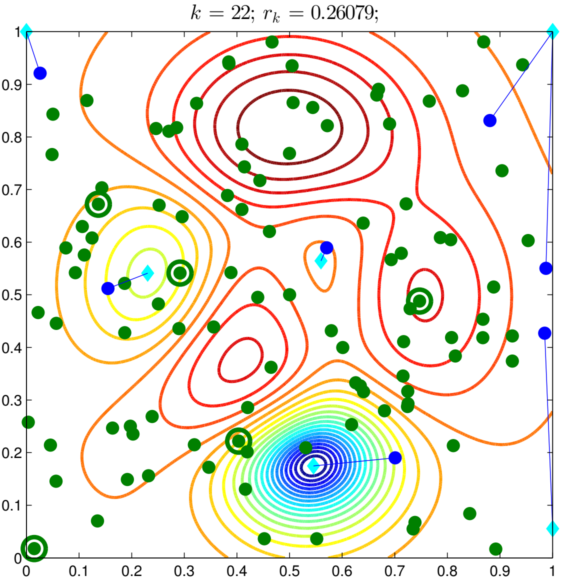
Multi-Level Single Linkage



Multi-Level Single Linkage



Multi-Level Single Linkage



Multi-Level Single Linkage

- $f \in C^2$, with local minima in the interior of $\mathcal{D}$, and the distance between these minima is bounded away from zero.
- $\mathcal{L}$ is strictly descent and converges to a minimum (not a stationary point).
- $$r_k = \frac{1}{\sqrt{\pi}} \sqrt[n]{\Gamma\left(1 + \frac{n}{2}\right) \mathrm{vol}\left(\mathcal{D}\right) \frac{\sigma \log kN}{kN}} \tag{1}$$

Theorem

If $r_k \to 0$, all local minima will be found almost surely.

Multi-Level Single Linkage

- $f \in C^2$, with local minima in the interior of $\mathcal{D}$, and the distance between these minima is bounded away from zero.
- $\mathcal{L}$ is strictly descent and converges to a minimum (not a stationary point).
- $$r_k = \frac{1}{\sqrt{\pi}} \sqrt[n]{\Gamma\left(1 + \frac{n}{2}\right) \mathrm{vol}\left(\mathcal{D}\right) \frac{\sigma \log kN}{kN}} \qquad (1)$$

Theorem

If $r_k \to 0$, all local minima will be found almost surely.

If r_k is defined by (1) with $\sigma > 4$, even if the sampling continues forever, the total number of local searches started is finite almost surely.

BAMLM

MLSL: (S2)–(S4)

$\hat{x} \in \mathcal{S}_k$

(S2) $\nexists x \in \mathcal{S}_k$ *with* $[\|\hat{x} - x\| \leq r_k$ *and* $f(x) < f(\hat{x})]$

(S3) $\hat{x}$ *has not started a local optimization run*

(S4) $\hat{x}$ *is at least* μ *from* $\partial\mathcal{D}$ *and* ν *from known local minima*

BAMLM

MLSL: (S2)–(S4)

$\hat{x} \in \mathcal{S}_k$

(S1) $\nexists x \in \mathcal{L}_k$ with $[\|\hat{x} - x\| \leq r_k$ and $f(x) < f(\hat{x})]$

(S2) $\nexists x \in \mathcal{S}_k$ *with* $[\|\hat{x} - x\| \leq r_k$ *and* $f(x) < f(\hat{x})]$

(S3) $\hat{x}$ *has not started a local optimization run*

(S4) $\hat{x}$ *is at least* μ *from* $\partial\mathcal{D}$ *and* ν *from known local minima*

BAMLM: (S1)–(S4), (L1)–(L6)

$\hat{x} \in \mathcal{L}_k$

(L1) $\nexists x \in \mathcal{L}_k$ $[\|\hat{x} - x\| \leq r_k$ and $f(x) < f(\hat{x})]$

(L2) $\nexists x \in \mathcal{S}_k$ with $[\|\hat{x} - x\| \leq r_k$ and $f(x) < f(\hat{x})]$

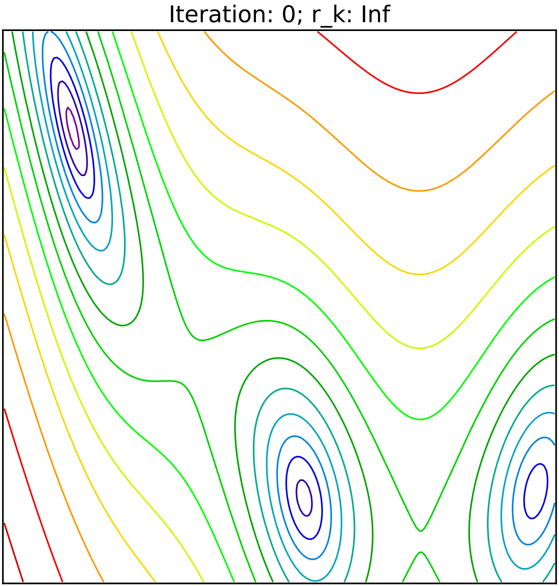
(L3) $\hat{x}$ has not started a local optimization run

(L4) $\hat{x}$ is at least μ from $\partial\mathcal{D}$ and ν from known local minima

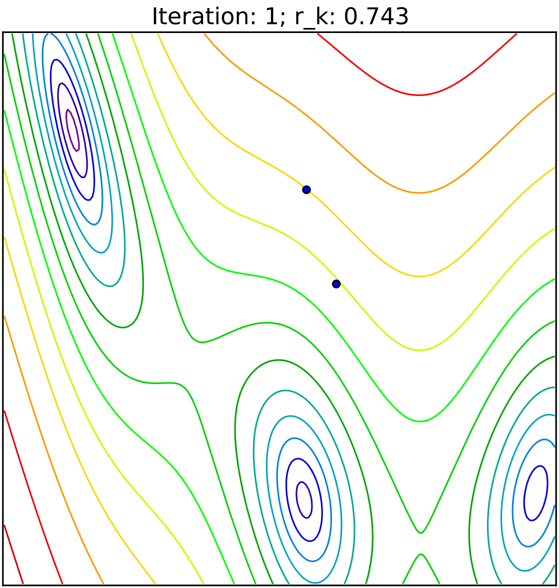
(L5) $\hat{x}$ is not in an active local optimization run and has not been ruled stationary

(L6) $\exists r_k$-descent path in $\mathcal{H}_k$ from some $x \in \mathcal{S}_k$ satisfying (S2-S4) to $\hat{x}$

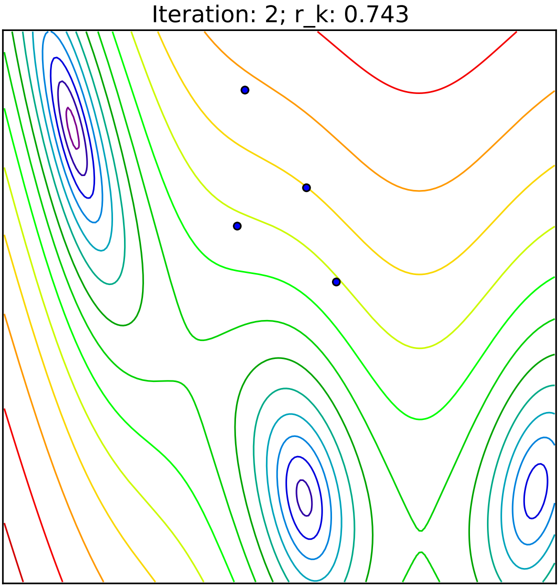
BAMLM



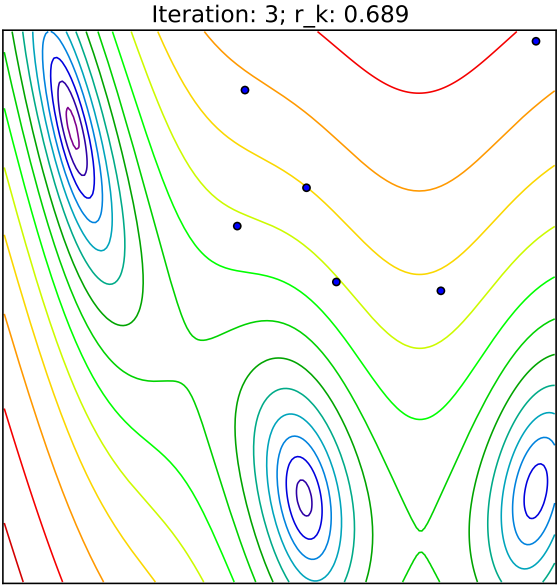
BAMLM



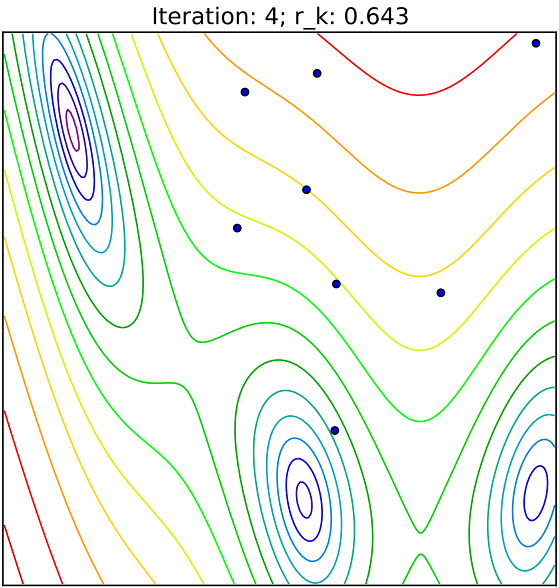
BAMLM



BAMLM

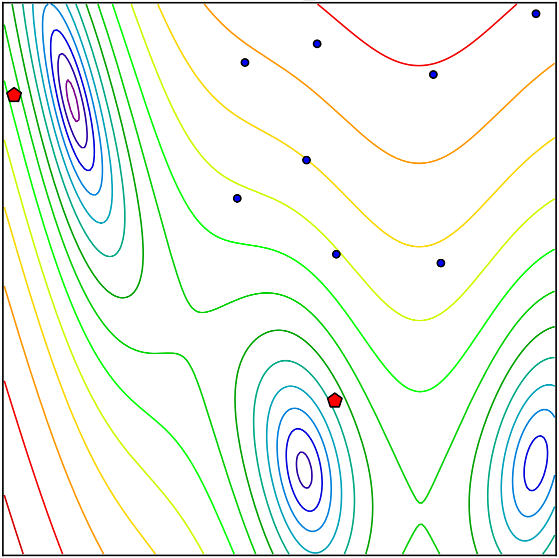


BAMLM

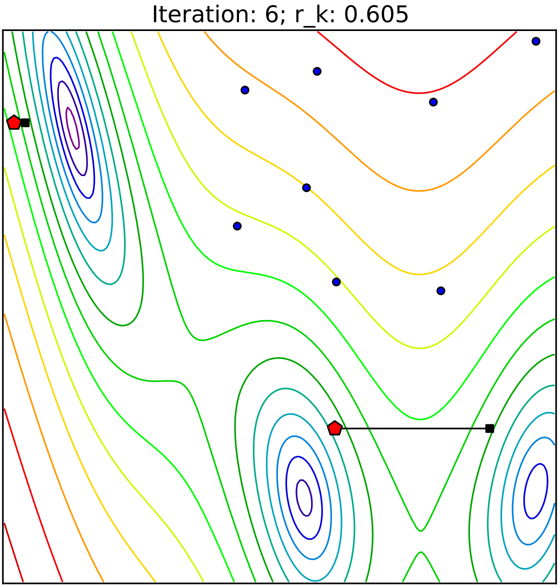


BAMLM

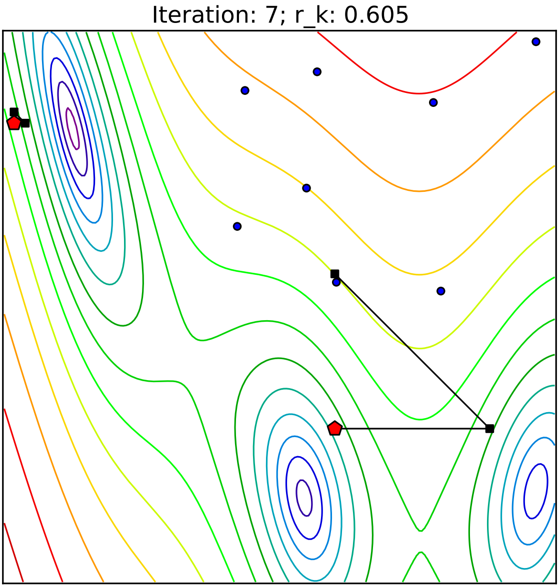
Iteration: 5; r_k: 0.605



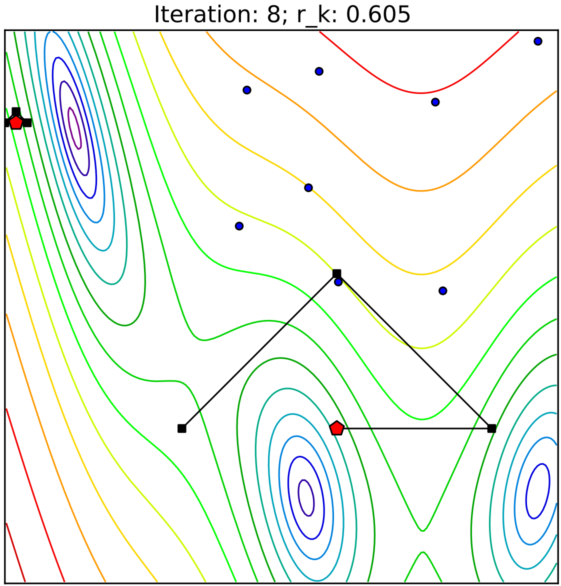
BAMLM



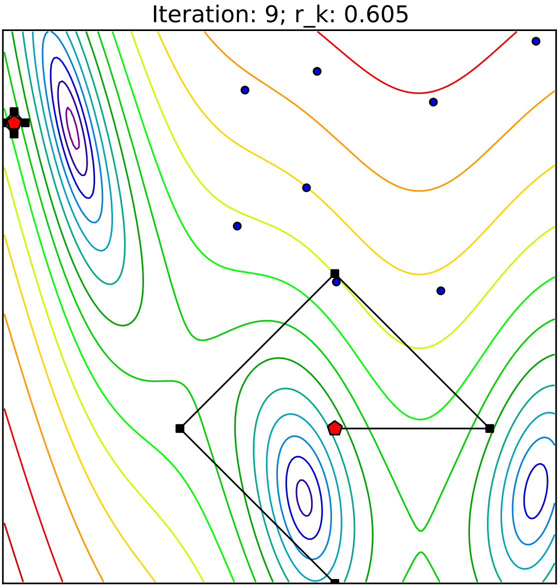
BAMLM



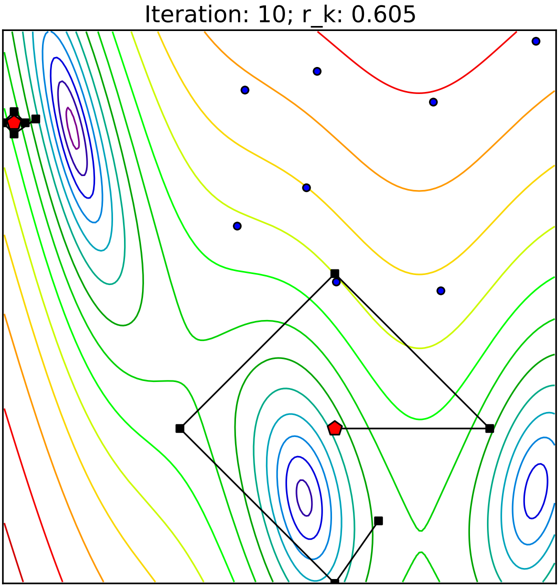
BAMLM



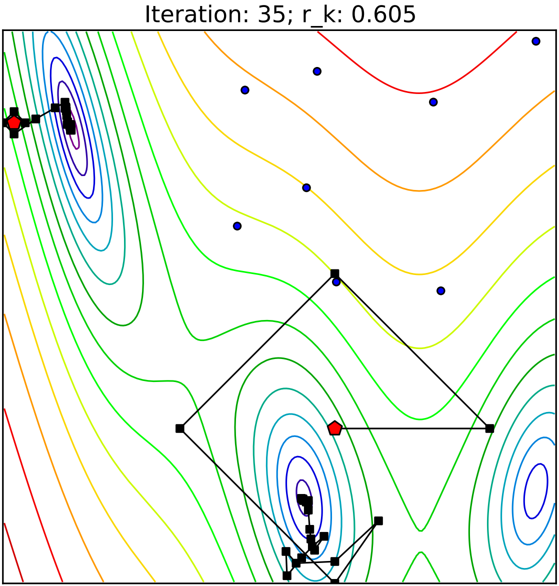
BAMLM



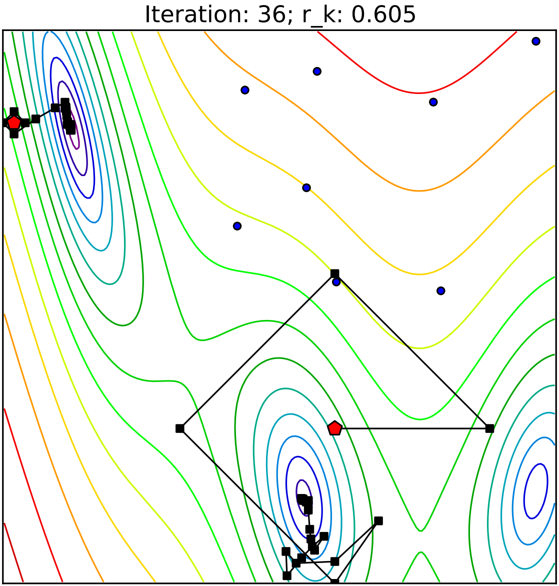
BAMLM



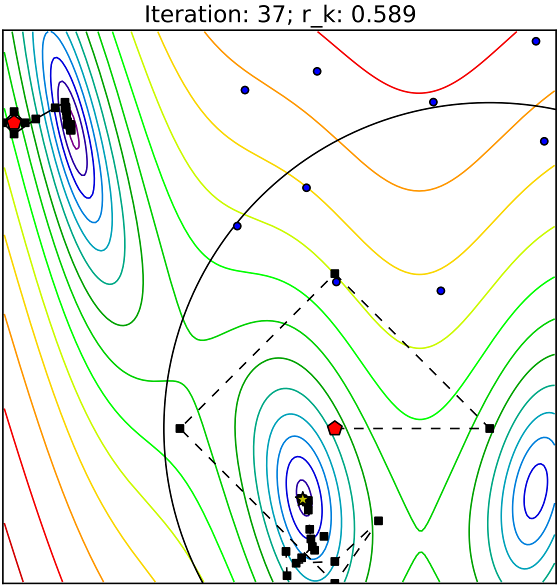
BAMLM



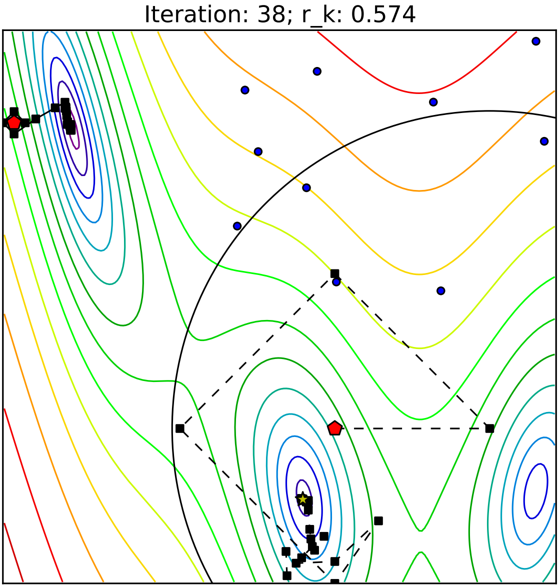
BAMLM



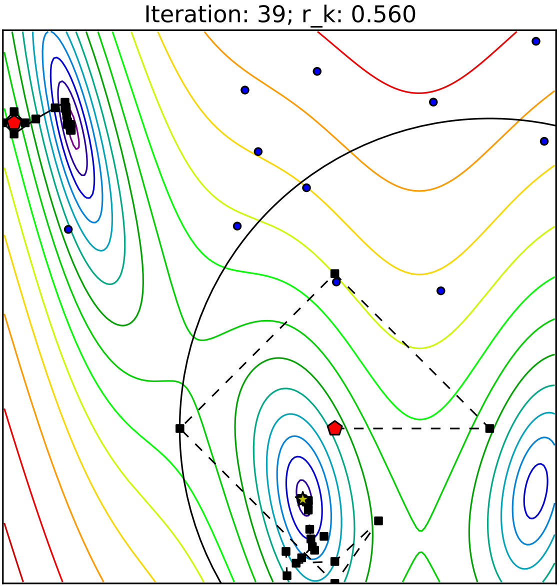
BAMLM



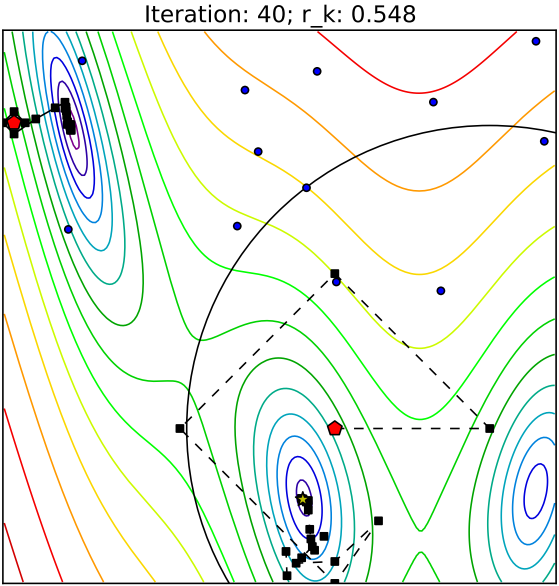
BAMLM



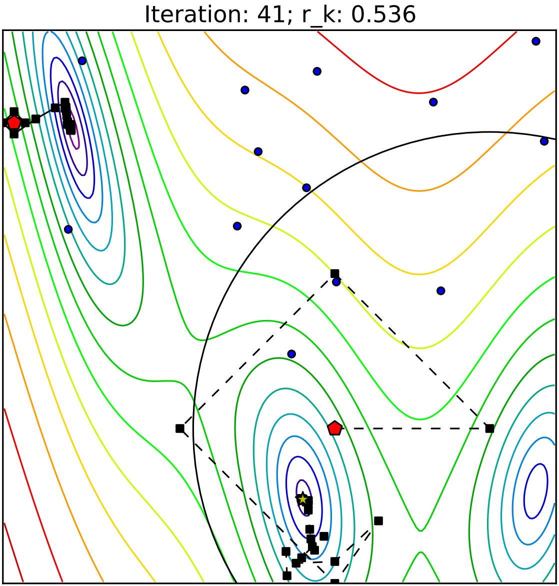
BAMLM



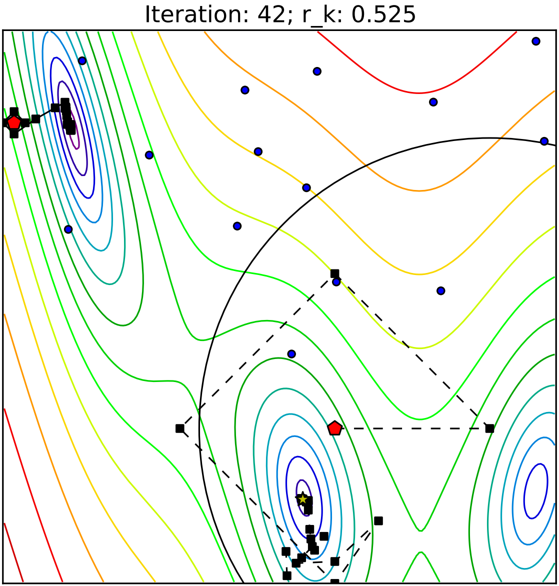
BAMLM



BAMLM

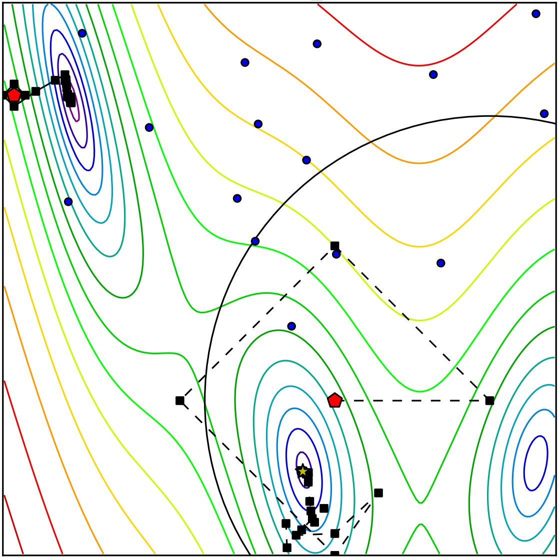


BAMLM

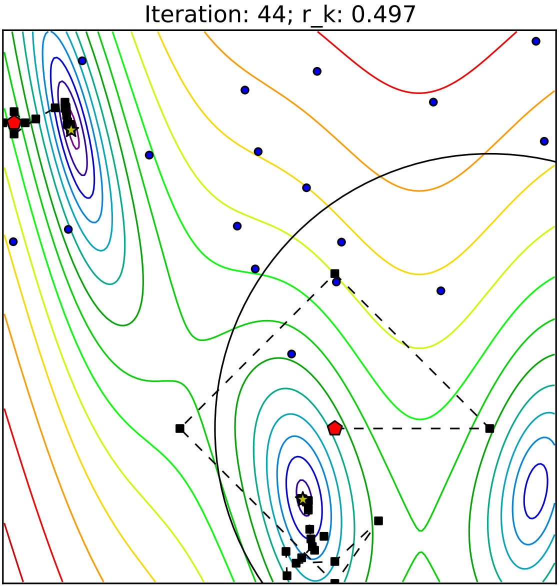


BAMLM

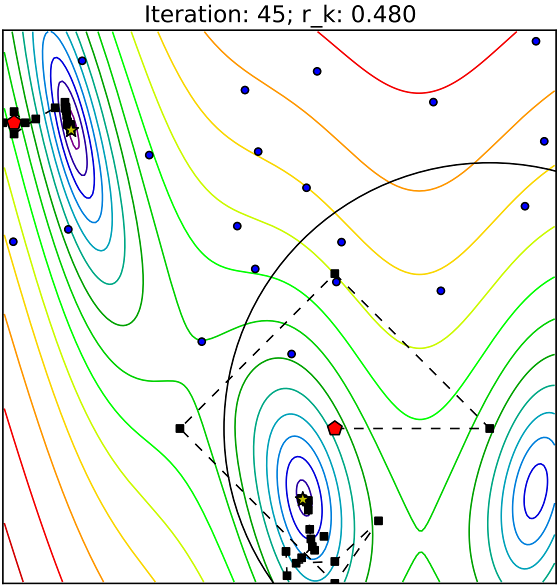
Iteration: 43; r_k: 0.515



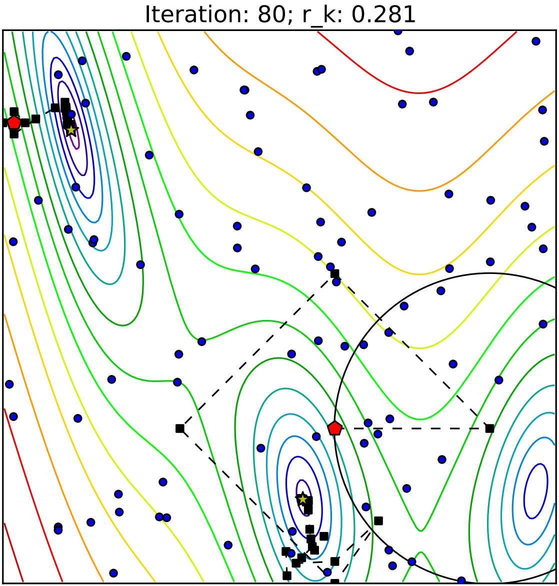
BAMLM



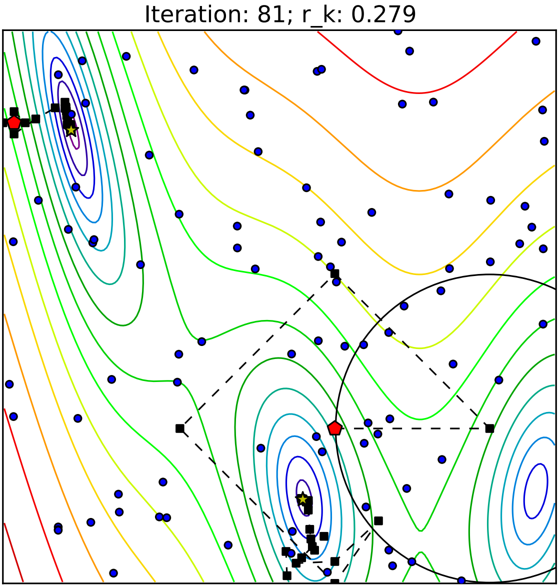
BAMLM



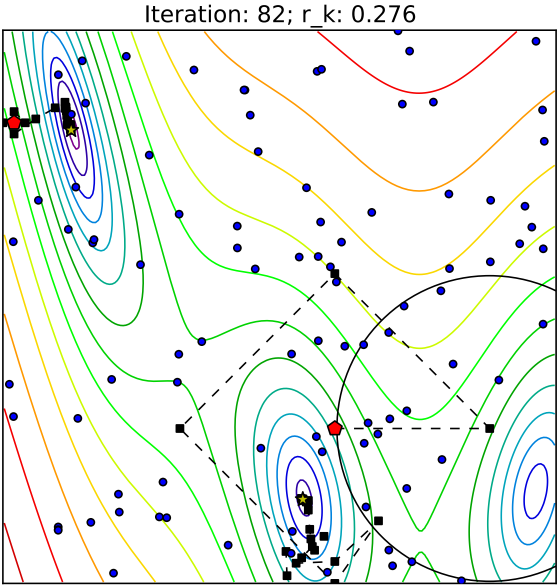
BAMLM



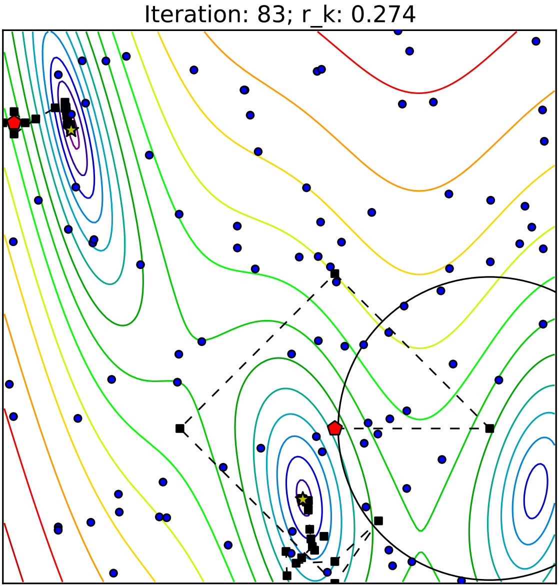
BAMLM



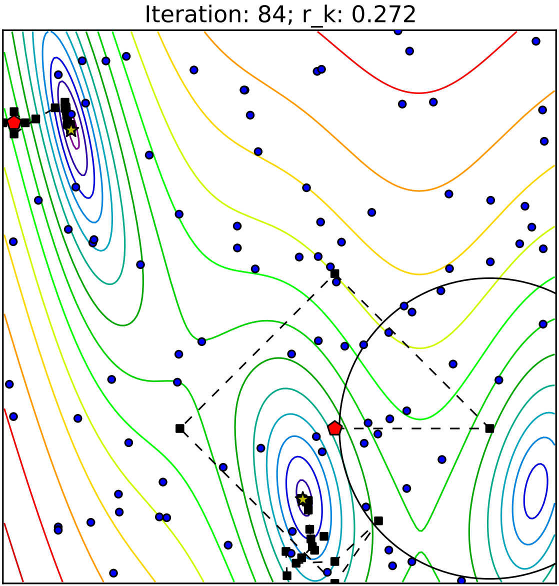
BAMLM



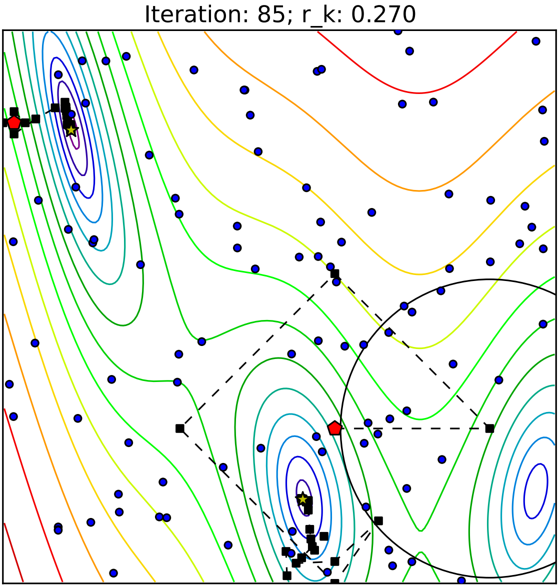
BAMLM



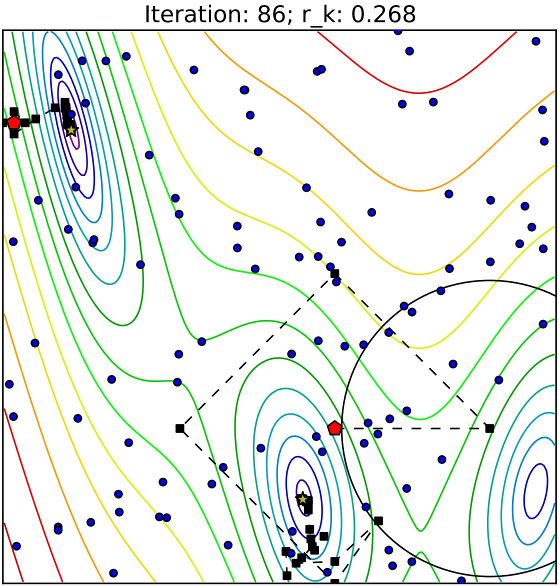
BAMLM



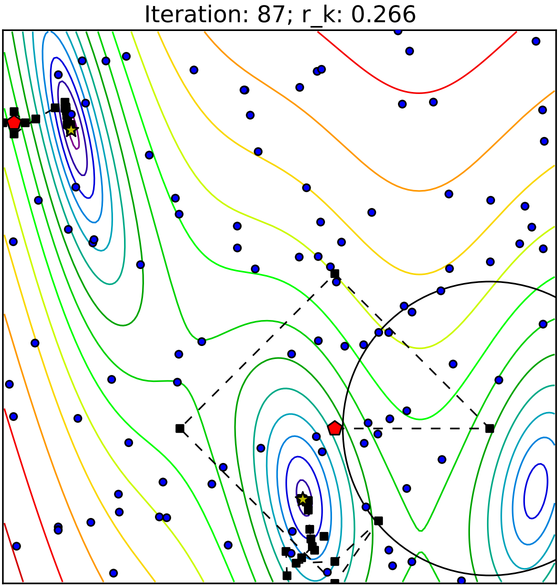
BAMLM



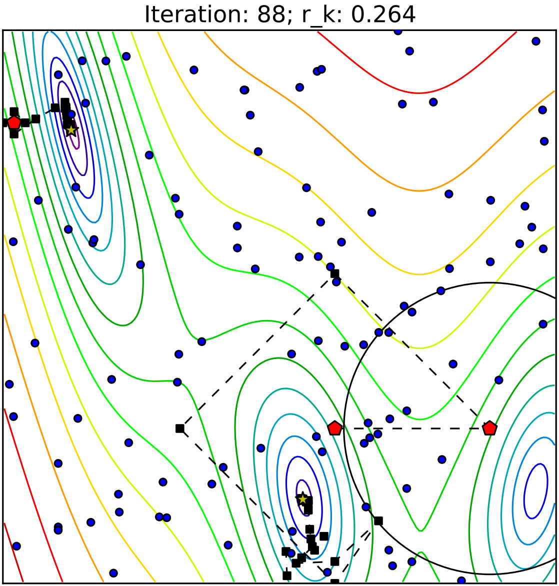
BAMLM



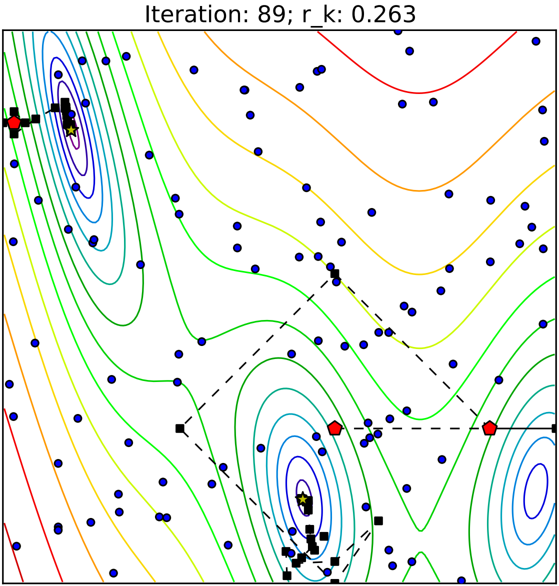
BAMLM



BAMLM

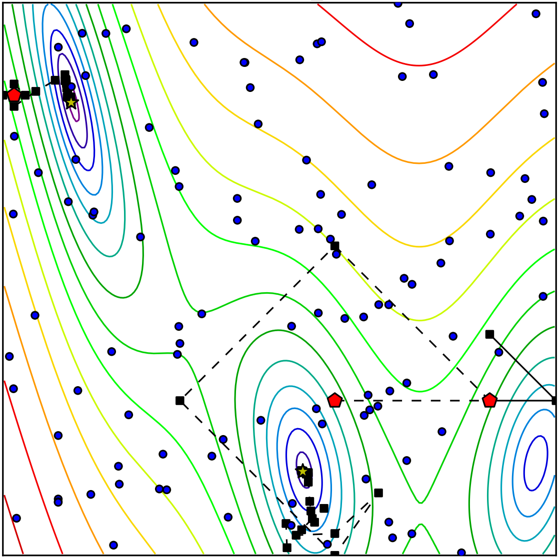


BAMLM

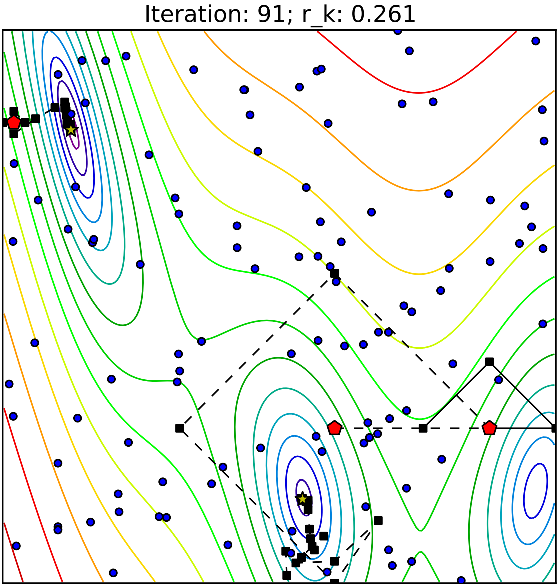


BAMLM

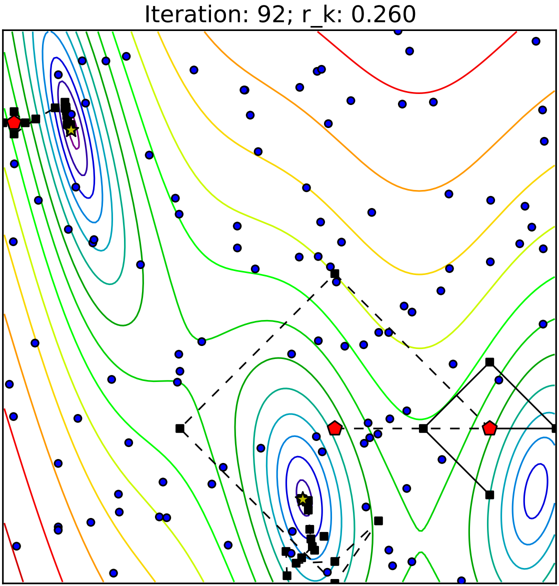
Iteration: 90; r_k: 0.262



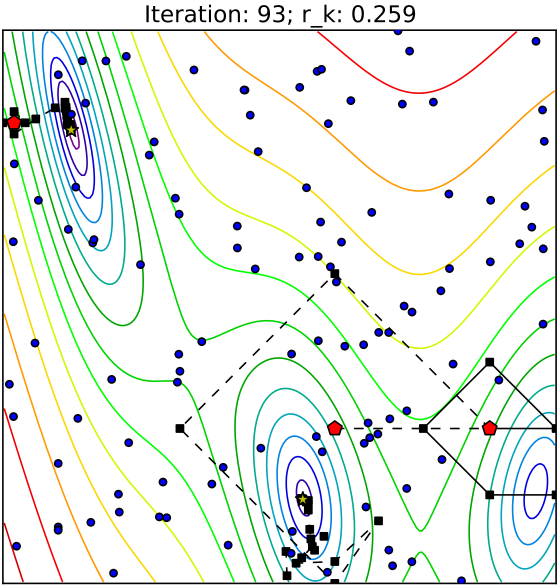
BAMLM



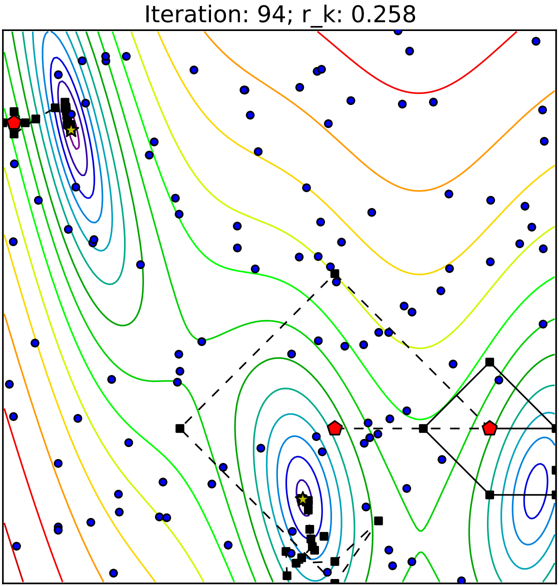
BAMLM



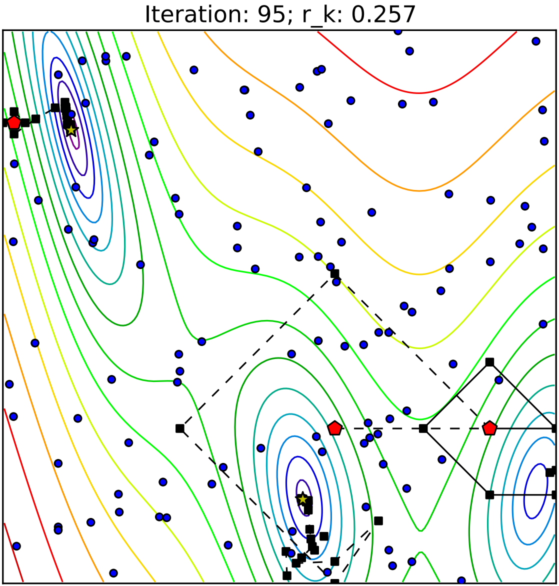
BAMLM



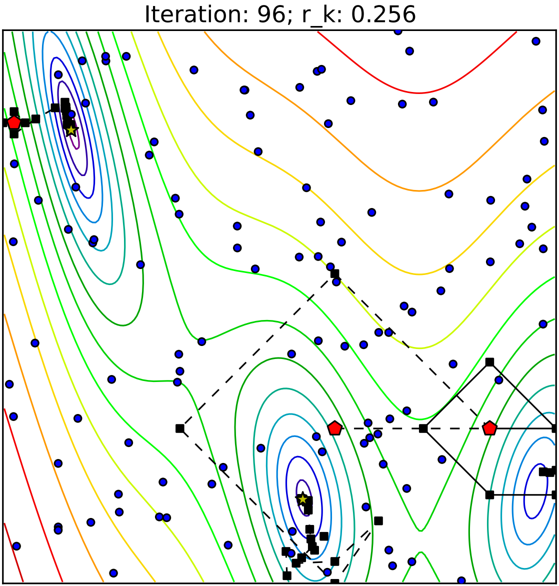
BAMLM



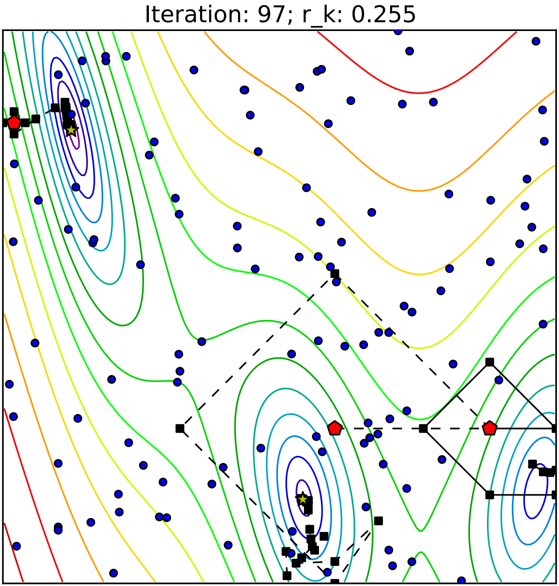
BAMLM



BAMLM

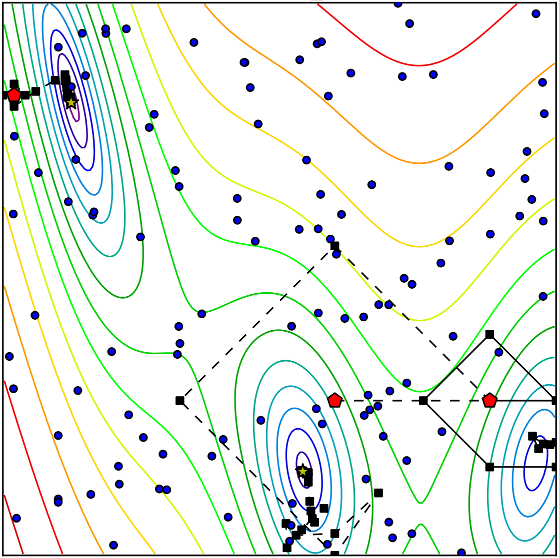


BAMLM



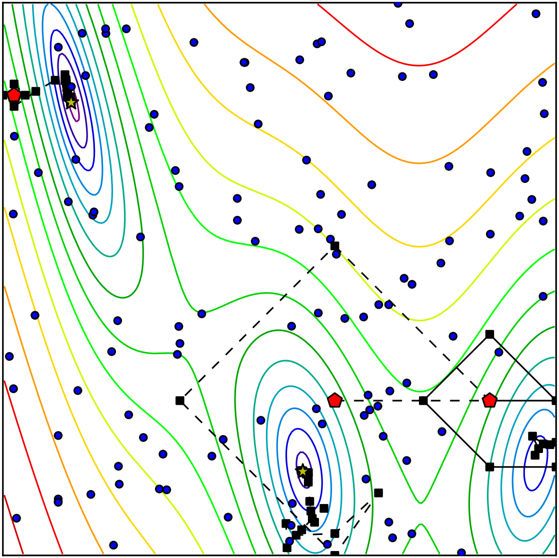
BAMLM

Iteration: 98; r_k: 0.255



BAMLM

Iteration: 99; r_k: 0.254



Properties of the local optimization method

Necessary:

- Honors a starting point
- Honors bound constraints

Properties of the local optimization method

Necessary:

- Honors a starting point
- Honors bound constraints

ORBIT satisfies these [Wild, Regis, Shoemaker, SIAM-JOSC, 2008]

BOBYQA satisfies these [Powell, 2009]

Properties of the local optimization method

Necessary:

- Honors a starting point
- Honors bound constraints

ORBIT satisfies these [Wild, Regis, Shoemaker, SIAM-JOSC, 2008]

BOBYQA satisfies these [Powell, 2009]

Possibly beneficial:

- Can return multiple points of interest
- Reports solution quality/confidence at every iteration
- Can avoid certain regions in the domain
- Uses a history of past evaluations of f
- Uses additional points mid-run

AAMLM

Algorithm 3: AAMLM

Give each worker a point to evaluate
for $k = 1, 2, \ldots$ **do**
  Receive from (longest waiting) worker w that has evaluated f
  Update $\mathcal{H}_k$ and r_k
  if *point evaluated by* w *is from an active run* **then**
    if *Run is complete* **then**
      Update X_k^*, and mark points inactive
    else
      Add the next point in its localopt run (not in $\mathcal{H}_k$) to Q_L
    Start run(s) at all point(s) satisfying (S1)–(S4), (L1)–(L6)
    Add the subsequent point (not in $\mathcal{H}_k$) from each run to Q_L
  Merge runs in Q_L with candidate minima within 2ν of each other
  Give w a point at which to evaluate f, either from Q_L or $\mathcal{R}$

BAMLM

MLSL: (S2)–(S4)

$\hat{x} \in \mathcal{S}_k$

(S1) $\nexists x \in \mathcal{L}_k$ with $[\|\hat{x} - x\| \leq r_k$ and $f(x) < f(\hat{x})]$

(S2) $\nexists x \in \mathcal{S}_k$ *with* $[\|\hat{x} - x\| \leq r_k$ *and* $f(x) < f(\hat{x})]$

(S3) $\hat{x}$ *has not started a local optimization run*

(S4) $\hat{x}$ *is at least* μ *from* $\partial\mathcal{D}$ *and* ν *from known local minima*

BAMLM: (S1)–(S4), (L1)–(L6)

$\hat{x} \in \mathcal{L}_k$

(L1) $\nexists x \in \mathcal{L}_k$ $[\|\hat{x} - x\| \leq r_k$ and $f(x) < f(\hat{x})]$

(L2) $\nexists x \in \mathcal{S}_k$ with $[\|\hat{x} - x\| \leq r_k$ and $f(x) < f(\hat{x})]$

(L3) $\hat{x}$ has not started a local optimization run

(L4) $\hat{x}$ is at least μ from $\partial\mathcal{D}$ and ν from known local minima

(L5) $\hat{x}$ is not in an active local optimization run and has not been ruled stationary

(L6) $\exists r_k$-descent path in $\mathcal{H}_k$ from some $x \in \mathcal{S}_k$ satisfying (S2-S4) to $\hat{x}$

AAMLM Theory

Theorem

Given the same assumptions as MLSL, AAMLM will start a finite number of local optimization runs with probability 1.

AAMLM Theory

Theorem

Given the same assumptions as MLSL, AAMLM will start a finite number of local optimization runs with probability 1.

Assumption

There exists $K_0 < \infty$ so that for any K_0 consecutive iterations, there is a positive (bounded away from zero) probability of evaluating a point from the sample stream and each existing local optimization run.

AAMLM Theory

Theorem

Given the same assumptions as MLSL, AAMLM will start a finite number of local optimization runs with probability 1.

Assumption

There exists $K_0 < \infty$ so that for any K_0 consecutive iterations, there is a positive (bounded away from zero) probability of evaluating a point from the sample stream and each existing local optimization run.

Theorem

Each $x^* \in X^*$ will almost surely be either identified in a finite number of evaluations or have a single local optimization run that is converging asymptotically to it.

Measuring Performance

GLODS Global & local optimization using direct search [Custódio, Madeira (JOGO, 2014)]

Direct Serial DIRECT [D. Finkel's MATLAB code]

pVTDirect Parallel DIRECT [He, Watson, Sosonkina (TOMS, 2009)]

Random Uniform sampling over domain (as a baseline)

BAMLM

- Concurrency: 4
- Local optimization method
  - ORBIT [Wild, Regis, & Shoemaker (SIAM JOSC, 2008)]
  - BOBYQA [Powell, 2009]
- Initial sample size: $10n$

- Each method evaluates Direct's $2n + 1$ initial points.

Measuring Performance

Notation:

Let X^* be the set of all local minima of f.
Let $f^*_{(i)}$ be the ith smallest value $\{f(x^*) | x^* \in X^*\}$.
Let $x^*_{(i)}$ be the element of X^* corresponding to the value $f^*_{(i)}$.

The global minimum has been found at a level $\tau > 0$ at batch k if an algorithm it has found a point $\hat{x}$ satisfying:

$$f(\hat{x}) - f^*_{(1)} \leq (1 - \tau) \left(f(x_0) - f^*_{(1)} \right),$$

where x_0 is the starting point for problem p.

Measuring Performance

Notation:

Let X^* be the set of all local minima of f.
Let $f^*_{(i)}$ be the ith smallest value $\{f(x^*) | x^* \in X^*\}$.
Let $x^*_{(i)}$ be the element of X^* corresponding to the value $f^*_{(i)}$.

The j best local minima have been found at a level $\tau > 0$ at batch k if:

$$\left|\left\{x^*_{(1)}, \dots, x^*_{(\underline{j}-1)}\right\} \bigcap \left\{x^*_{(i)} : \exists x \in \mathcal{H}_k \text{ with } \left\|x - x^*_{(i)}\right\| \leq r_n(\tau)\right\}\right| = \underline{j} - 1$$

$$\&$$

$$\left|\left\{x^*_{(\underline{j})}, \dots, x^*_{(\bar{j})}\right\} \bigcap \left\{x^*_{(i)} : \exists x \in \mathcal{H}_k \text{ with } \left\|x - x^*_{(i)}\right\| \leq r_n(\tau)\right\}\right| \geq j - \underline{j} + 1,$$

where $\underline{j}$ and $\bar{j}$ are the smallest and largest integers such that $f^*_{(\bar{j})} = f^*_{(j)} = f^*_{(\underline{j})}$ and where $r_n(\tau) = \sqrt[n]{\frac{\tau \operatorname{vol}(\mathcal{D})\Gamma(\frac{n}{2}+1)}{\pi^{n/2}}}$.

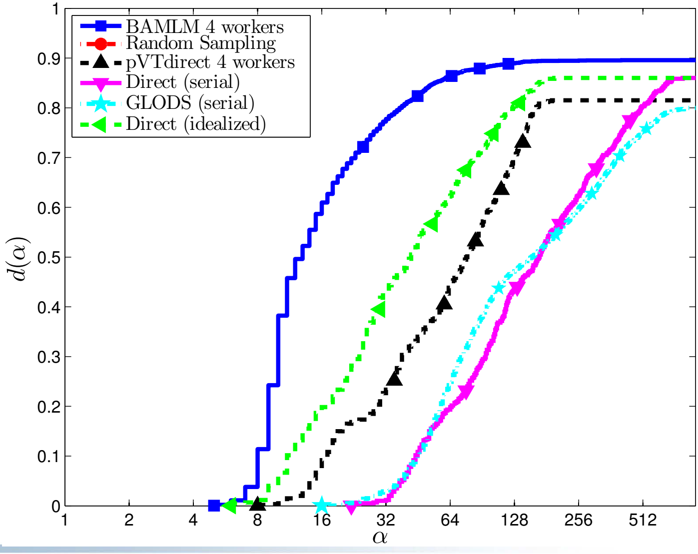
Problems considered

GKLS problem generator [Gaviano et al., “Algorithm 829” (TOMS, 2003)]

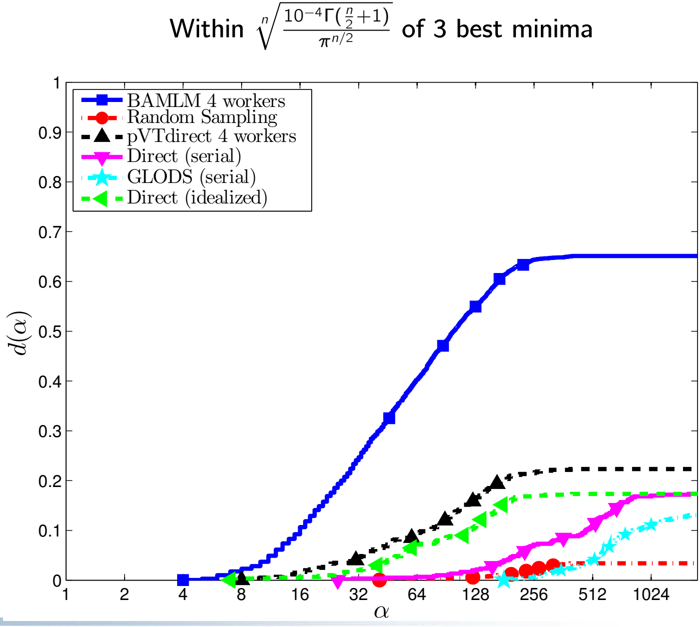
- 600 synthetic problems with known local minima
- $n = 2, \ldots, 7$
- 10 local minima in the unit cube with a unique global minimum
- 100 problems for each dimension
- 5 replications (different seeds) for each problem
- 5000 evaluations

Data Profiles

$$f(x) - f^*_{(1)} \leq (1 - 10^{-5}) \left(f(x_0) - f^*_{(1)} \right)$$

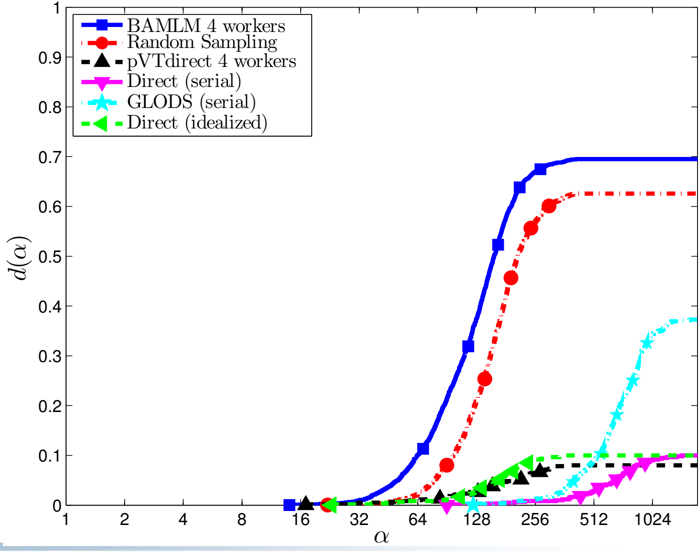


Data Profiles



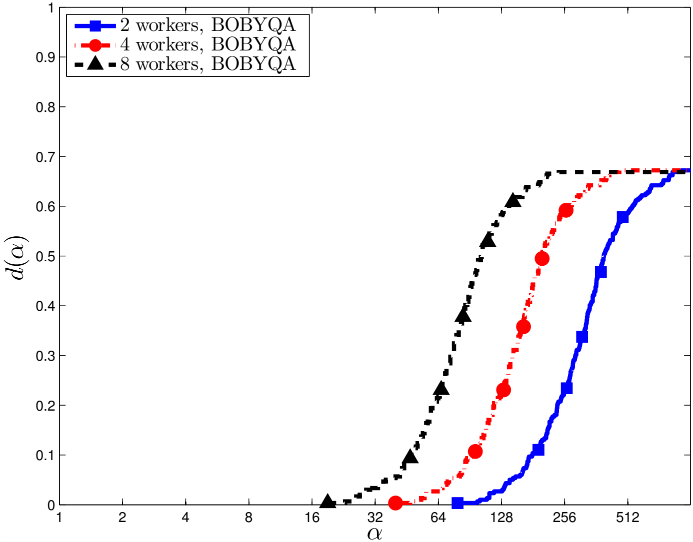
Data Profiles

Within $\sqrt[n]{\frac{10^{-3}\Gamma(\frac{n}{2}+1)}{\pi^{n/2}}}$ of 7 best minima



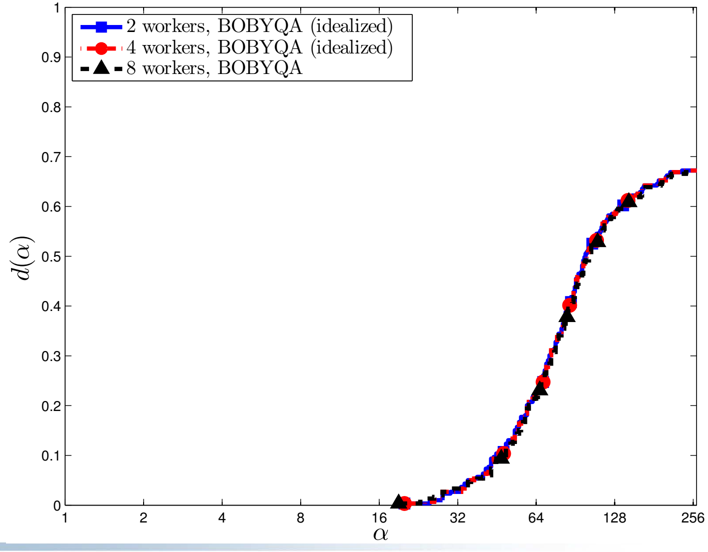
Data Profiles

Within $\sqrt[n]{\frac{10^{-3}\Gamma(\frac{n}{2}+1)}{\pi^{n/2}}}$ of 7 best minima



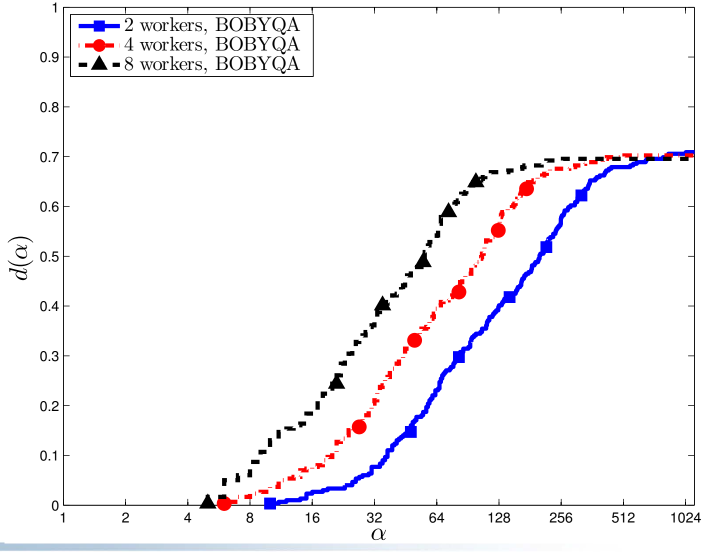
Data Profiles

Within $\sqrt[n]{\frac{10^{-3}\Gamma(\frac{n}{2}+1)}{\pi^{n/2}}}$ of 7 best minima



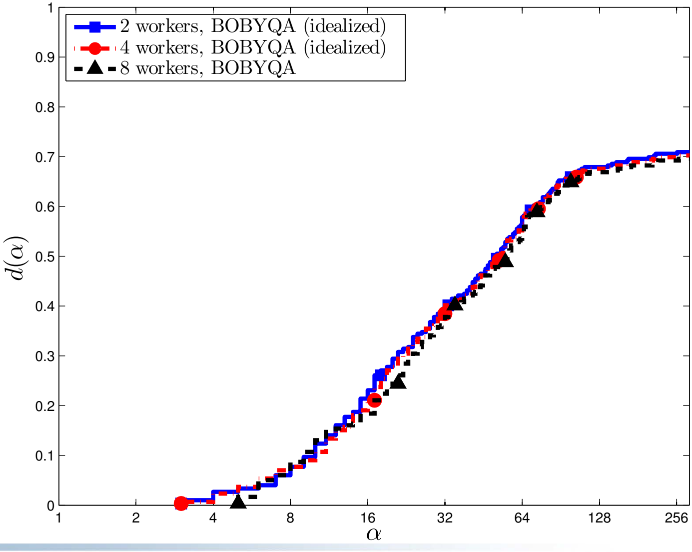
Data Profiles

Within $\sqrt[n]{\frac{10^{-4}\Gamma(\frac{n}{2}+1)}{\pi^{n/2}}}$ of 3 best minima



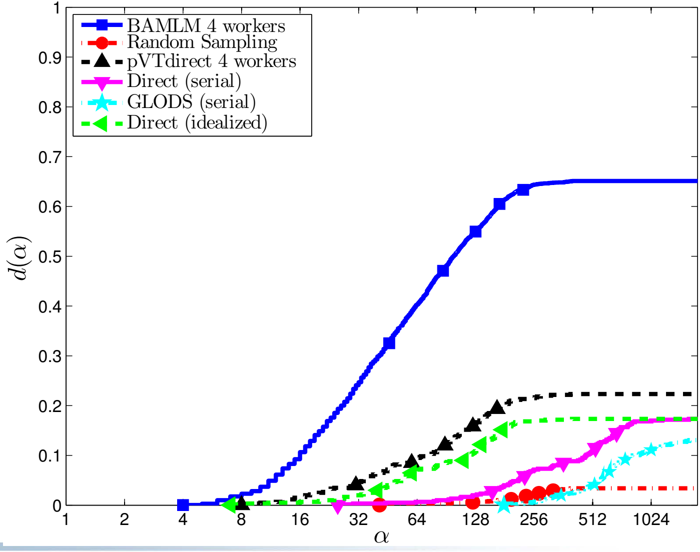
Data Profiles

Within $\sqrt[n]{\frac{10^{-4}\Gamma(\frac{n}{2}+1)}{\pi^{n/2}}}$ of 3 best minima



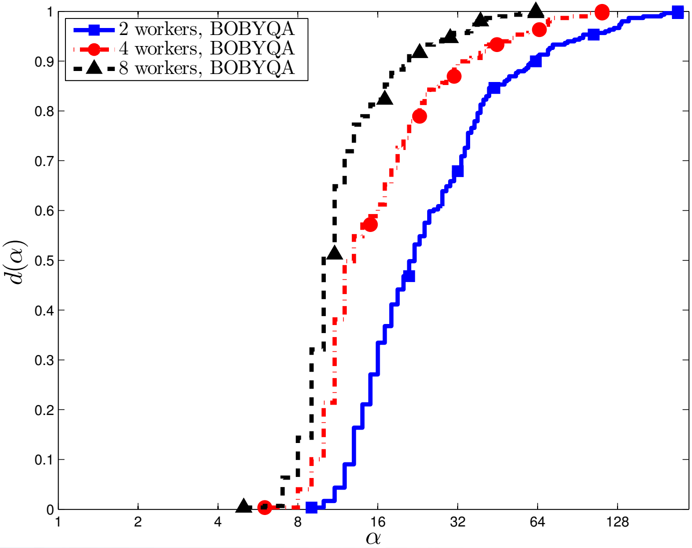
Data Profiles

Within $\sqrt[n]{\frac{10^{-4}\Gamma(\frac{n}{2}+1)}{\pi^{n/2}}}$ of 3 best minima



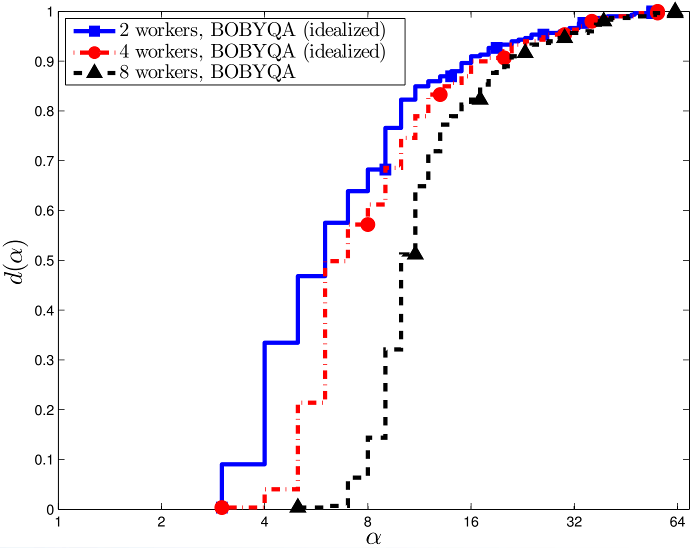
Data Profiles

$$f(x) - f^*_{(1)} \leq (1 - 10^{-5}) \left(f(x_0) - f^*_{(1)} \right)$$

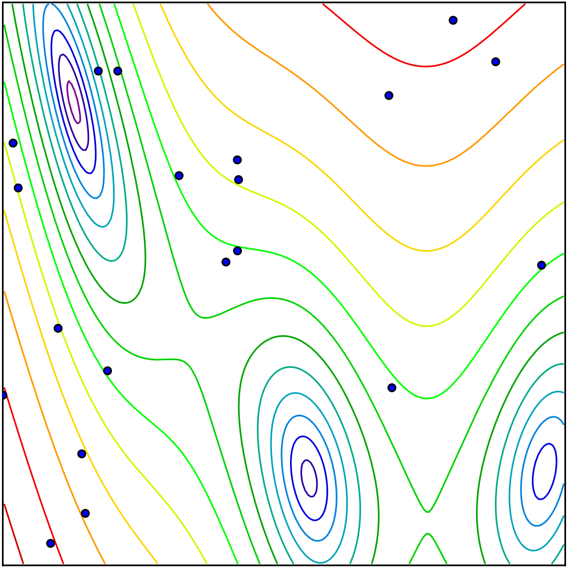


Data Profiles

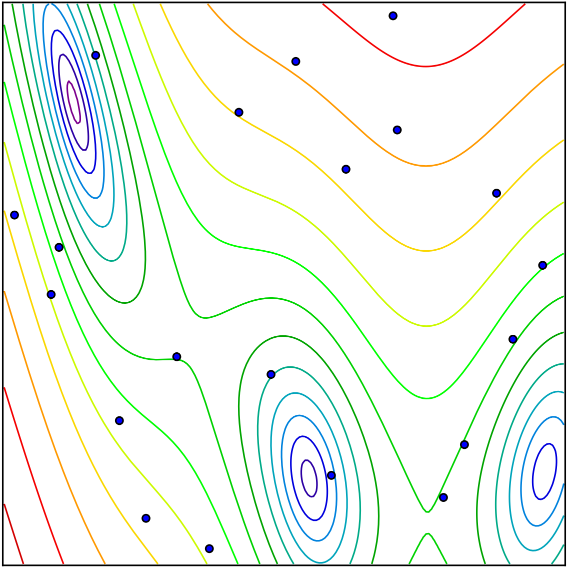
$$f(x) - f^*_{(1)} \leq (1 - 10^{-5}) \left(f(x_0) - f^*_{(1)} \right)$$



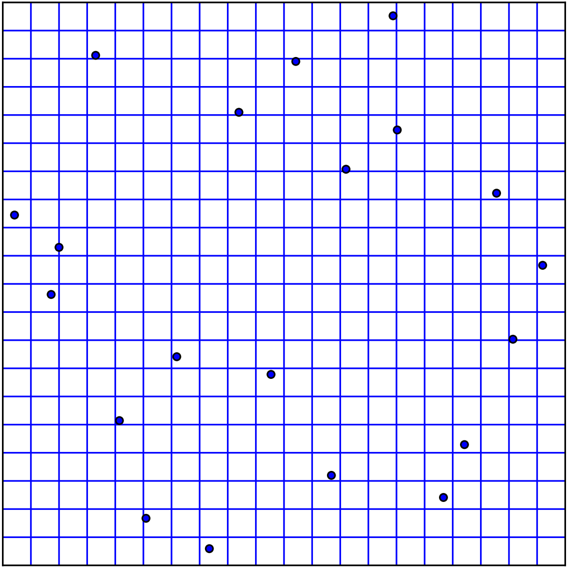
Uniform sampling



Latin hypercube sampling



Latin hypercube sampling



BAMLM with LHS

Critical distance for uniform sampling:

$$r_k = \pi^{-1/2} \left(\Gamma(1 + \frac{n}{2}) \operatorname{vol}(\mathcal{D}) \frac{\sigma \log kN}{kN} \right)^{1/n}$$

Critical distance for Latin hypercube sampling:

$$r_k = \pi^{-1/2} \left(\Gamma(1 + \frac{n}{2}) \operatorname{vol}(\mathcal{D}) \frac{\sigma N^{n-1} \log k}{k} \right)^{1/n} \tag{2}$$

BAMLM with LHS

Critical distance for uniform sampling:

$$r_k = \pi^{-1/2} \left(\Gamma(1 + \frac{n}{2}) \,\mathrm{vol}\,(\mathcal{D})\, \frac{\sigma \log kN}{kN} \right)^{1/n}$$

Critical distance for Latin hypercube sampling:

$$r_k = \pi^{-1/2} \left(\Gamma(1 + \frac{n}{2}) \,\mathrm{vol}\,(\mathcal{D})\, \frac{\sigma N^{n-1} \log k}{k} \right)^{1/n} \quad (2)$$

Theorem

If r_k is defined by (2) with $\sigma > 4$, even if the sampling continues forever, the total number of local runs started by BAMLM (or AAMLM) is finite almost surely.

Does LHS help?

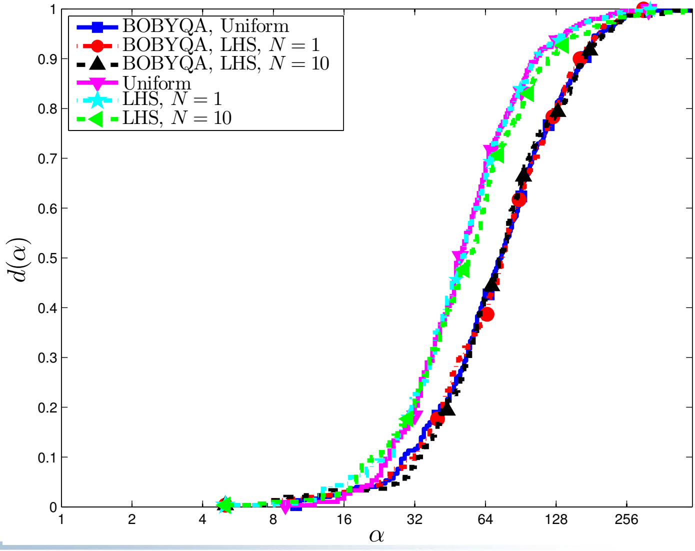
Problem setup:

- 10 different GKLS problems
- 5 different seeds
- $n = 2, \ldots, 7$

- Same starting LHS sample of $10n$ points (except for uniform)
- Same (uniform) r_k value

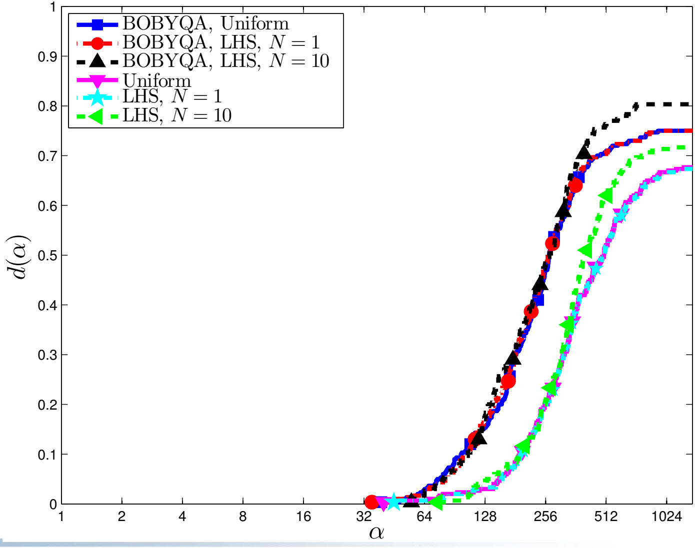
Data Profiles

Within $\sqrt[n]{\frac{10^{-2}\Gamma(\frac{n}{2}+1)}{\pi^{n/2}}}$ of 5 best minima



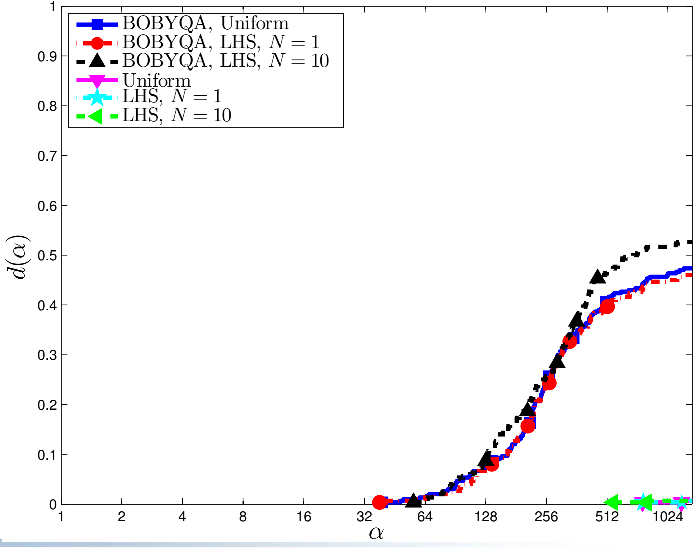
Data Profiles

Within $\sqrt[n]{\frac{10^{-3}\Gamma(\frac{n}{2}+1)}{\pi^{n/2}}}$ of 5 best minima



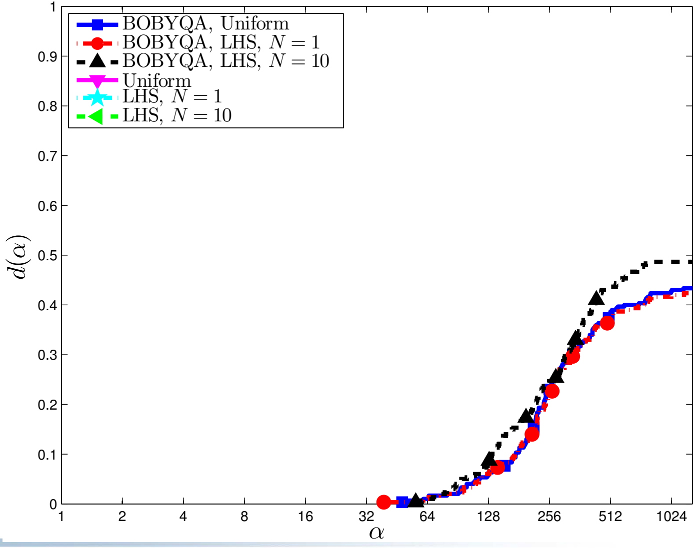
Data Profiles

Within $\sqrt[n]{\frac{10^{-4}\Gamma(\frac{n}{2}+1)}{\pi^{n/2}}}$ of 5 best minima



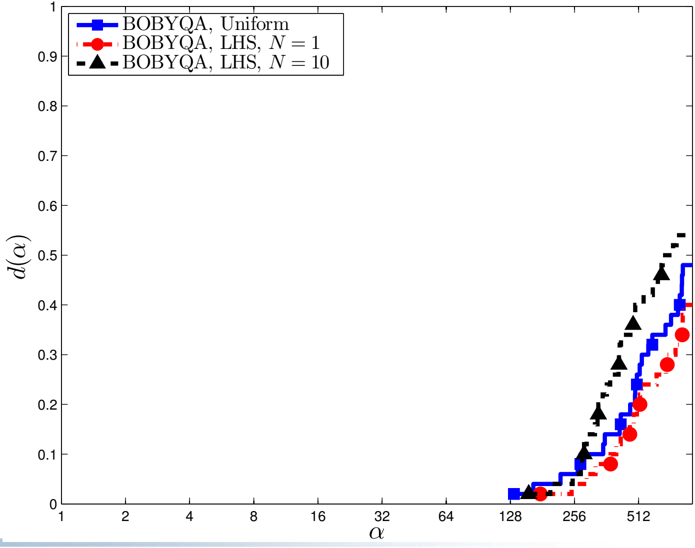
Data Profiles

Within $\sqrt[n]{\frac{10^{-5}\Gamma(\frac{n}{2}+1)}{\pi^{n/2}}}$ of 5 best minima



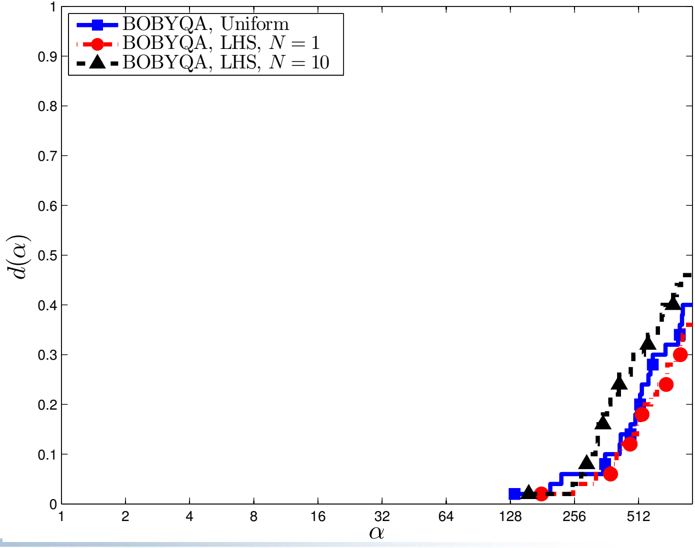
Data Profiles

Within $\sqrt[n]{\frac{10^{-4}\Gamma(\frac{n}{2}+1)}{\pi^{n/2}}}$ of 5 best minima, $n=6$



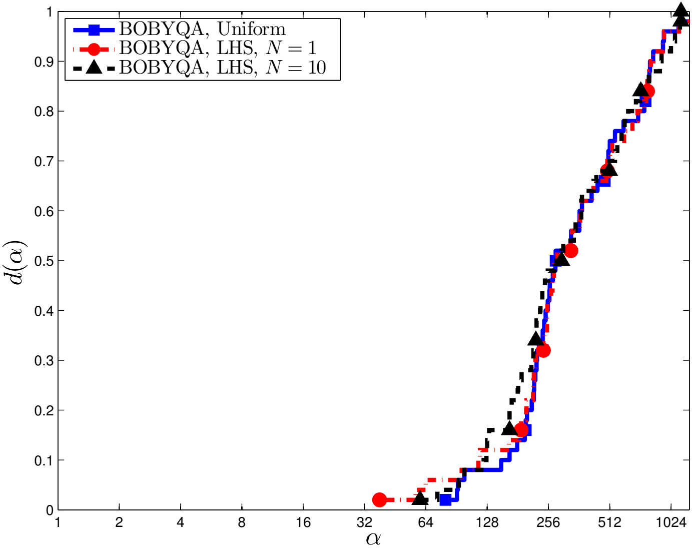
Data Profiles

Within $\sqrt[n]{\frac{10^{-5}\Gamma(\frac{n}{2}+1)}{\pi^{n/2}}}$ of 5 best minima, $n=6$



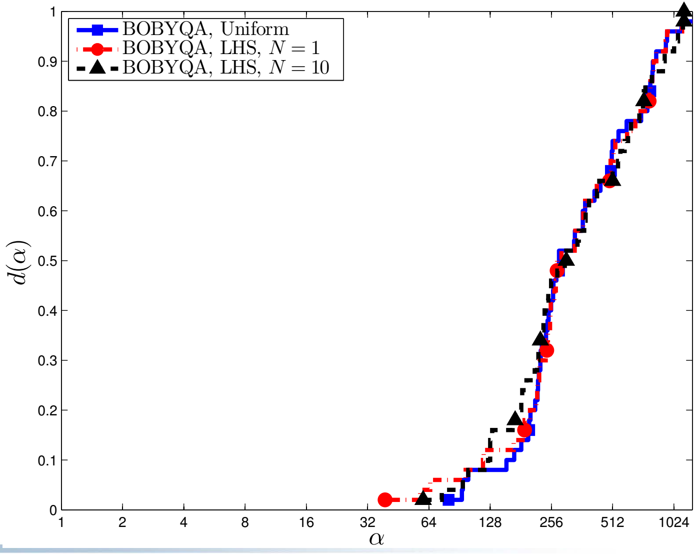
Data Profiles

Within $\sqrt[n]{\frac{10^{-4}\Gamma(\frac{n}{2}+1)}{\pi^{n/2}}}$ of 5 best minima, $n = 3$



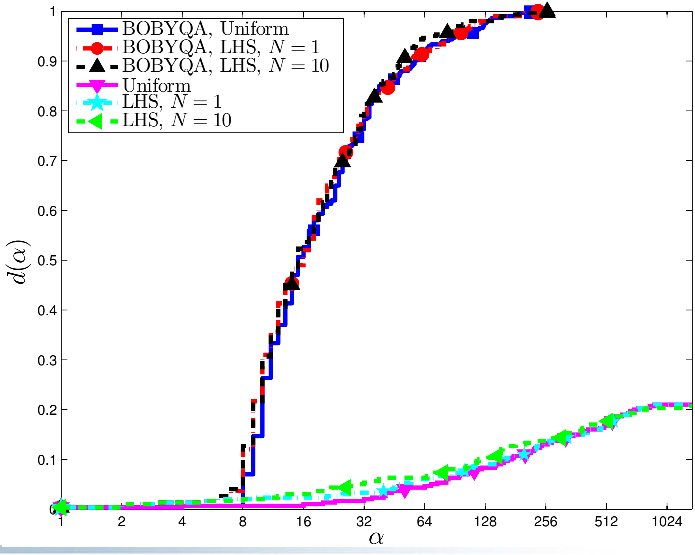
Data Profiles

Within $\sqrt[n]{\frac{10^{-5}\Gamma(\frac{n}{2}+1)}{\pi^{n/2}}}$ of 5 best minima, $n = 3$



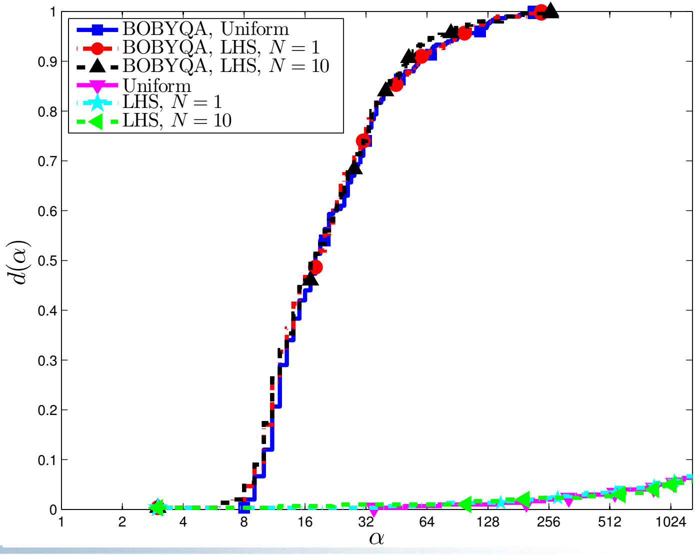
Data Profiles

$$f(x) - f^*_{(1)} \leq (1 - 10^{-2}) \left(f(x_0) - f^*_{(1)} \right)$$



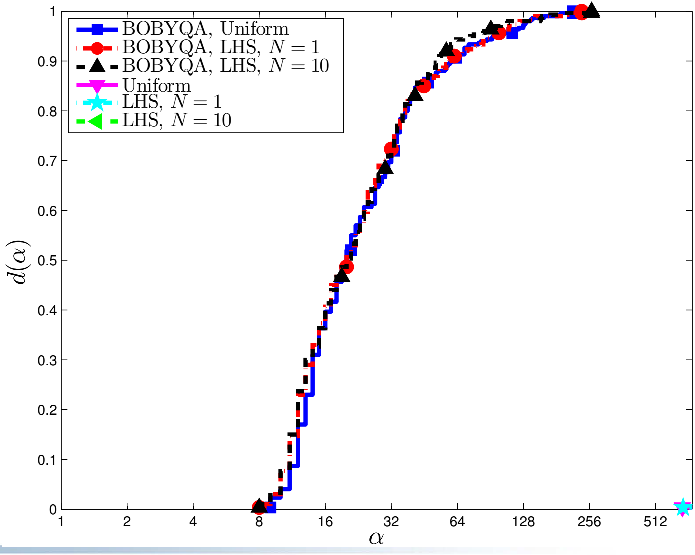
Data Profiles

$$f(x) - f^*_{(1)} \leq (1 - 10^{-3})\left(f(x_0) - f^*_{(1)}\right)$$



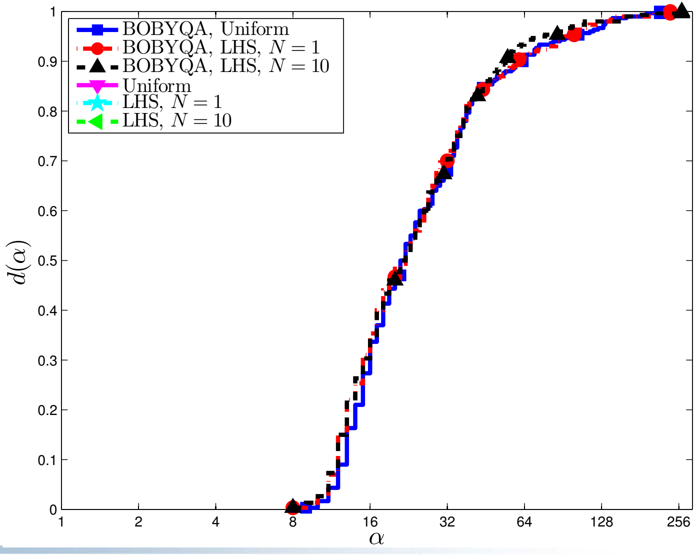
Data Profiles

$$f(x) - f^*_{(1)} \leq (1 - 10^{-4}) \left(f(x_0) - f^*_{(1)} \right)$$



Data Profiles

$$f(x) - f^*_{(1)} \leq (1 - 10^{-5}) \left(f(x_0) - f^*_{(1)} \right)$$



Closing Remarks

- Concurrent function evaluations can locate multiple minima while efficiently finding the global minimum.

Closing Remarks

- Concurrent function evaluations can locate multiple minima while efficiently finding the global minimum.

- Latin hypercube sampling appears to help find more minima in higher-dimensional problems.

Questions:

- Finding (or designing) the best local solver for our framework?
- Best way to process the queue?

AAMLM

Algorithm 3: AAMLM

Give each worker a point to evaluate
for $k = 1, 2, \ldots$ **do**
    Receive from (longest waiting) worker w that has evaluated f
    Update $\mathcal{H}_k$ and r_k
    if *point evaluated by* w *is from an active run* **then**
        if *Run is complete* **then**
            Update X_k^*, and mark points inactive
        else
            Add the next point in its localopt run (not in $\mathcal{H}_k$) to Q_L
        Start run(s) at all point(s) satisfying (S1)–(S4), (L1)–(L6)
        Add the subsequent point (not in $\mathcal{H}_k$) from each run to Q_L
    Merge runs in Q_L with candidate minima within 2ν of each other
    Give w a point at which to evaluate f, either from Q_L or $\mathcal{R}$